Higher Order Statistics of channel capacity in κ-μ fading channel

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***Abstract*—The frequency scarcity imposed by the fast-growing need for mobile data service requires promising spectrum aggregation systems. The so-called higher-order statistics (HOS) of the channel capacity (CC) is a suitable metric on the system performance. While prior relevant works have improved our knowledge of HOS characterization on the spectrum aggregation systems, an analytical framework encompassing generalized fading models of interest is not yet available. However, the expressions of HOS are not correct in several previous research works. In this paper, we present novel method by expressing the closed-form expression of CC as the sum of weighted exponential terms and then invoke multinomial expansion to obtain the required coefficients and utilize MGF (Moment Generating Function) based maximum ratio combining (MRC) diversity receivers technique over κ-μ fading distribution to compute higher order moments. Also, we provide correct, simplified and efficient HOS expressions for the asymptotically low and high signal-to-noise regimes and provide a detailed HOS analysis of κ-μ fading channel by obtaining vital statistical measures, such as the amount of dispersion, skewness, and kurtosis by the HOS results. Finally, all derived expressions are validated via the Semi-infinite Gauss Hermite quadrature method.**

***Index Terms*—exponential-type approximation for channel capacity, prony's approximation, higher-order statistics, κ-μ generalized fading distribution, multinomial theorem**

I. Introduction

For the last six decades, the researchers are searching for generalized techniques in computing the typical metric for performance evaluation, i.e., channel capacity (CC) of the fading channels. We find that the first-order statistics (FOS) of the (CC) $\bar{C} = \mathbb{E}\left[\log\left(1+\gamma\right)\right]$ for a certain averaged SNR; where, γ denotes the end-to-end instantaneous signal-to-noise ratio (SNR), $\mathbb{E}[\cdot]$ denotes the expectation operator, and $\log(\cdot)$ denotes the natural logarithm. also well-known as averaged channel capacity (ACC) or ergodic channel capacity and second-order statistics (SOS), i.e., the variance of the CC has been widely studied in the literature, considering different fading environments [1–7], and references therein.

Due to continually growing mobile data demand for future wireless communication systems, it becomes more and more difficult to allocate a wide and contiguous frequency band to each user equipment and base station has brought about increasing scarcity in the available radio spectrum. In the research of wireless communications, the higher-order statistics (HOS) of the channel capacity (CC) can fully explore the reliability of the signal transmission and can adequately explain the CC dispersion induced by the heterogeneity that inherently exists in spectrum aggregation systems [8]. Moreover, valuable insights into the spectrum aggregation implications on the transmission reliability can also be deduced by deriving the HOS of the CC.

Lately, many theoreticians, practitioners, and researchers [7, 9–17] directed their study on the HOS of CC to ensure the reliability of wireless transmissions and its quality. Despite its importance, [16] pointed out that HOS of the CC received relatively scant attention among the researchers and in the literature, due in part to the intractability of its analysis, especially compared to the first-order statistics.

The references in [9] discussed multiple-input multiple-output (MIMO) transmission over Rayleigh or Riciean fading channels; they also discuss the HOS of the CC only for single-link lognormal fading channels. Sagias et al. [10] showed the probability density function (PDF)- based framework is valid only for diversity combining receivers in Rayleigh and Nakagami- m fading environments and presented the HOS of the CC for several diversity receivers taking into account the effects of independent and non-identically distributed (i.n.i.d.) Nakagami-m fading channels. Later, in [11] the authors presented the first-moment generating function (MGF) based approach for the accurate HOS of CC analysis in fading environments such that it eliminates all difficulties that emerged from [7] and [10] investigated the HOS of the CC for amplify-and-forward (AF) multihop systems over gamma and generalized gamma fading channels. An example is a generic framework for the asymptotic HOS of the CC over independent and identically distributed (i.i.d.) Nakagami-m fading channels were provided in [11]. Also, an MGF-based approach for the HOS of the CC for L-branch MRC receivers has been proposed in [11] with an example application of correlated Nakagami-m fading channels. In particular, Yilmaz & Alouini proposed in [13] a moment generating function (MGF)-based approach for the ACC analysis, specifically introducing how to unify the ACC analyses of diversity combining and transmission schemes into a single MGF-based analysis. The article, [14] studied HOS for the CC of equal gain combining (EGC) Receivers Over Generalized Fading Channels. Moreover, fruitful insights into the implications of spectrum aggregation on transmission reliability can be extracted by deriving the HOS of the channel capacity.

Despite its importance, there is still a gap in the literature in calculating the HOS of the channel capacity correctly due to its analysis's intractability and hence received relatively little attention in the literature. Most of the final expressions involve complicated mathematical expressions and are not versatile enough to be applied to generalized fading channels. Most of the literature for the generalized fading models, the expression for calculating the higher-order statistics via MGF method shown in terms of the Fox H or the Meijer- G functions [11, 14, 16–20], which are computationally inefficient (their infinite series representations are also not very attractive for further manipulations). Moreover, more importantly, they are not providing any useful insights. Even when exact closed-form solutions exist, the complexity of this form often overshadows the elegance of the proposed solution. Furthermore, likewise, difficulties in evaluating it numerically. This shortcoming motivates the research for a solution that is more simple and elegant in deriving useful insights. The results beyond variance i.e. the equation for skewness and kurtosis are incorrectly stated in [11],[13], (Eq. 32 and Eq. 33 in [15]),[14, 21] and other subsequently published works. The conclusion drawn from these articles is highly incorrect. For example, the skewness is incorrectly stated as

$$\text{Skewness} = \frac{\mathbb{E}\left[C_3\right] - \mathbb{E}^3[C_1]}{\left(\mathbb{E}\left[C_2\right] - \mathbb{E}^2[C_1]\right)^{3/2}} \tag{1}$$

which should be

$$\text{Skewness} = \frac{\mathbb{E}\left[C_3\right] - 3\mathbb{E}[C_1]\mathbb{E}\left[C_2\right] + 2\mathbb{E}^3[C_1]}{\left(\mathbb{E}\left[C_2\right] - \mathbb{E}^2[C_1]\right)^{3/2}} \tag{2}$$

while the kurtosis, which is given by

$$\text{Kurtosis} = \frac{\mathbb{E}[C_4] - \mathbb{E}^4[C_1]}{\left(\mathbb{E}[C_2] - \mathbb{E}^2[C_1]\right)^2} \tag{3}$$

is incorrectly stated, which should be

$$\text{Kurtosis} = \frac{\mathbb{E}[C_4] - 4\mathbb{E}[C_1]\mathbb{E}[C_3] + 6\mathbb{E}^2[C_1]\mathbb{E}[C_2] - 3\mathbb{E}^4[C_1]}{\mathbb{E}[C_2] - \mathbb{E}^2[C_1]^2} \tag{4}$$

where, $\mathbb{E}$ is expectation operator
C_n = n^{th} moment of Channel Capacity

As a consequence, most of the trends extrapolated from these papers are incorrect. Also, we propose an exact closed-form solution (without using identities i.e., the Fox H or the Meijer- G functions) yet is simple, elegant, and likewise easy to evaluate HOS of the CC over generalized distributions where the exact closed-form solutions are ordinarily unattainable.

We develop several different new mathematical techniques to address some of the shortcomings. Our approximation could dramatically improve the accuracy and spectral efficiency of the communication system. Specifically, in this paper, $\kappa - \mu$ fading channel is considered as a generalized fading channel, which incorporates many exceptional cases such as the Rician, Nakagami-m, Rayleigh, and one-sided Gaussian distributions and the expressions for the HOS of the CC are presented in closed form. The above-mentioned performance metrics can efficiently help select the best channel in heterogeneous wireless networks with spectrum sharing and/or aggregation capabilities.

The main contribution of this paper are as follows.

1) We present an accurate and efficient technique for computing the desired HOS of CC with different types of diversity combining methods. This will dramatically simplify the calculation of CC since the n^{th} moments of the CC $\mathbb{E}[C_n] = [\log_2^n(1+\gamma)]$ can be expressed as the weighted sum of the moment generated function of the signal to noise ratio and then invoke the multinomial expansion prior averaging over the fading SNR statistics over $\kappa - \mu$ fading environment. Our approximation could dramatically improve the accuracy and spectral efficiency of the communication system. Specifically, in this paper, $\kappa - \mu$ fading channel is considered, as a generalized fading model, which incorporates many special cases such as the Rician, Nakagami-m, Rayleigh, and one-sided Gaussian distributions and the expressions for the HOS of the CC are presented in closed form. The performance metrics above-mentioned can efficiently help to select the best channel in heterogeneous wireless networks with spectrum sharing and/or aggregation capabilities.
2) The equation for skewness and kurtosis are incorrectly stated in [11],[13], (Eq. 32 and Eq. 33 in [15]),[14, 21]. The correct expressions for the same are presented in this work. Thus, the proposed contribution is meaningful, since it addresses to rectify the errors in computing the HOS of the CC over generalized fading channels.

The remainder of this paper is organized as follows: Sec. II describes the methodology. **??** considered system and channel model. Sec. III is devoted to the derivation of the HOS of the CC over $\kappa - \mu$ fading channels. The numerical results and analysis are provided in the respective sections, while concluding remarks are finally given in Sec. IV.

II. Methodology

A. Shannon Hartley's Theorem

In data communication channel capacity C (in bit per second) is the maximum rate at which data can be reliably transmitted in terms of bit per second over a communication channel. Shannon's theorem describes the maximum bit- rate that can be transmitted with an arbitrarily small bit-error rate (BER) with a given signal power over a channel with bandwidth B (in Hz) which is affected by additive white gaussian noise (AWGN). The Eq. 5 is the Shannon-Hartley Theorem.

$$C = B\log_2(1+\gamma) \tag{5}$$

where, C is the channel capacity in bit per second
B is the bandwidth in Hz
γ is the signal to noise ratio in terms of bit per second

B. Exponential - Type Approximations for $\log_2^n(1+\gamma)$

In [22], Olabiyi presented an approach to approximate the Gaussian probability integral $Q(.)$ using the exponential type of approximation. Borrowing this idea we approximate the values of the coefficients a_k and b_k for n = [1,2,3,4] and we have chosen N = [4,5,7,8,10,13,15,17,19] after an extensively researching for the right number of terms for the exponential type approximation of $\log_2^n(1+\gamma)$ as per Eq. 6 by splitting linear and nonlinear terms using least square error approximation (a) Least squared minimization of absolute error (AMSE) and (b) Least square minimization of relative error (RMSE).

$$\log_2^n(1+\gamma) \approxeq \sum_{k=1}^{N} a_{(k,n)} e^{-(b_{k,n})\gamma} \tag{6}$$

where, n = power, N is the number of terms
a_k and b_k are the linear and non-linear coefficients we are interested in approximating.

1) Splitting the Linear and Nonlinear Problems: MATLAB provides several functions for solving non-linear least squares problems. Older versions of MATLAB have one general=purpose, multidimensional non-linear minimizer, $fmins$. In more recent versions of MATLAB, $fmins$ has been updated, and its name changed to $fminsearch$. The Optimization Toolbox provides additional capabilities, including a minimizer for problems with constraints, $fmincon$, a minimizer for unconstrained problems,$fminunc$, and two functions intended specifically for non-linear least squares, $lsqnonlin$, and $lsqcurvefit$.

Notice that the fitting problem is linear in the parameters (a_1,a_2,$\cdots$,a_N). This means for any values of (b_1,b_2 ,$\cdots$,b_N), we can use the backslash operator to find the values of (a_1,a_2,$\cdots$,a_N) that solve the least-squares problem. We find that choosing a bad starting point for the curve fitting using the normal $lsqcurvefit$ is a trivial task as it leads to a local solution that is not globalchoosing a starting point with the same bad (b_1,b_2,$\cdots$,b_N) values for the split two-parameter problem leads to the global solution.

Separable least squares curve fitting problems involve both linear and non-linear parameters. We could ignore the linear portion and use $lsqcurvefit$ to search for all the parameters. However, if we take advantage of the separable structure, We can rework the problem as a two-dimensional problem and can search for the best values of non-linear parameters (b_1,b_2,$\cdots$,b_N). A split problem is a more efficient and robust technique to initial guess. With this approach, $lsqcurvefit$ is used to search for values of the non-linear parameters that minimize the residual norm. The values of linear parameters (a_1,a_2,$\cdots$,a_N) are calculated at each step using the backslash operator.

2) Least squared minimization of absolute error (AMSE): When n=1, the R.H.S of the Eq. 6 can be rewritten as $\hat{y}_i$ in the Eq. 7 to carry out error minimization. N on the L.H.S. of the Eq. 6 is the number of terms used to approximate. The Eq. 8 is the mean square criterion where, w is the sample data size.

$$\hat{y}_i = \sum_{k=1}^{N} a_k e^{-\lambda k x_i} = a_1 e^{-b_1 x_i} + \cdots + a_N e^{-b_N x_i} \tag{7}$$

$$MSE = \sum_{i=1}^{w} (y_i - \hat{y}_i)^2 \tag{8}$$

Since the dependence of $\hat{y}$ on the coefficients a_k is linear, we may split the above nonlinear data fitting problem into linear and nonlinear forms to improve robustness against an improper selection of initial conditions (global vs. local convergence).

Let us look at the linear optimization problem, To find the optimum coefficients for the linear inverse problem (given initial parameters $b_1, b_2, \cdots, b_N$), we differentiate Eq. 9 w.r.t a_k and set it to 0.

$$\frac{dMSE}{da_k} = -2\sum_{i=1}^{w}(y_i - \sum_{k=1}^{N}(a_k e^{-b_k x_i}))e^{-b_k x_i} \tag{9}$$

The Eq. 10 is in the vector form as,

$$\underset{\substack{\uparrow \\ ydata}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_w \end{bmatrix}_{w\times 1}} = \underset{\substack{\uparrow \\ T}}{\begin{bmatrix} e^{-b_1 x_1} & \cdots & e^{-b_N x_1} \\ e^{-b_2 x_2} & \cdots & e^{-b_N x_2} \\ \vdots & \ddots & \vdots \\ e^{-b_w x_w} & \cdots & e^{-b_N x_w} \end{bmatrix}_{w\times N}} \times \underset{\substack{\uparrow \\ S}}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}_{N\times 1}}$$

with w linear parameters S, and w nonlinear parameters b_w. To solve for the linear parameters C, we build a matrix A where the N-th column of T is $e^{(-b_N \times xdata)}$, where, $xdata$ is a vector. Then we solve $A \times S$ = $ydata$ for the linear least-squares solution S, where $ydata$ is the observed values of $\hat{y}_i$.

$$S = [T]^{-1} \times ydata \tag{10}$$

3) Least squared minimization of relative error (RMSE): To minimize the relative error, we have

$$MSE = \left[\frac{\sum_{i=1}^{w}(y_i - \sum_{k=1}^{N}(a_k e^{-b_k x_i}))}{y_i}\right]^2 \tag{11}$$

Now after we differentiate Eq. 11 w.r.t a_k and set it to 0 we get Eq. 12.

$$\frac{dMSE}{da_k} = \frac{-2e^{-b_k x_i}}{y_i}\sum_{i=1}^{w}(1 - \frac{\sum_{k=1}^{N}(a_k e^{-b_k x_i})}{y_i}) \tag{12}$$

Thus,

$$\underset{\substack{\uparrow \\ ydata}}{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{w\times 1}} = \underset{\substack{\uparrow \\ P}}{\begin{bmatrix} \frac{e^{-b_1 x_1}}{y_1} & \cdots & \frac{e^{-b_N x_1}}{y_1} \\ \frac{e^{-b_2 x_2}}{y_2} & \cdots & \frac{e^{-b_N x_2}}{y_2} \\ \vdots & \ddots & \vdots \\ \frac{e^{-b_w x_w}}{y_w} & \cdots & \frac{e^{-b_N x_w}}{y_w} \end{bmatrix}_{w\times N}} \times \underset{\substack{\uparrow \\ Q}}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}_{N\times 1}}$$

$$Q = [P]^{-1} \times ydata \tag{13}$$

with w linear parameters Q, and w nonlinear parameters b_w. To solve for the linear parameters C, we build a matrix P where the N-th column of P is $e^{(-b_N \times xdata)}$, where, $xdata$ is a vector. Then we solve $P \times Q$ = $ydata$ for the linear least-squares solution Q, where $ydata$ is the observed values of $\hat{y}_i$.

4) Curve-fitting: In order to perform the curve fit on method 1 and method 2, we select 4000 data points uniformly distributed between the ranges $-1 \leq x \leq 3$, and subsequently define $\gamma = 10^x$ (i.e.) 4000 non-uniformly distributed data points in the linear scale. x=$-1 : 0.001 : 3$ $\gamma = 0.1 \rightarrow 1000$, n = Power, and N = Number of terms.

5) Multinomial Expansion: For n = 1, we recommend to use initial guess as, x = [4e-14 13e-4 26e-4 85e-4 97e-4 17e-3 33e-3 66e-3 118e-3 12e-2 23e-2 4e-1 8e-1 1.46 1.47 2.35 2.84 2.93 3.05]. We obtain this seed value by using multinomial expansion with n = 2 and N=7 which gives us 28 coefficients for $a_{(k,n)}$ and $b_{(k,n)}$ to start. For n = [2,3,4] and and N = [4,5,7,8,10,13,15,17,19], we use multinomial expansion theorem on Eq. 14 to get number of terms to get the coefficients $a_{(k,n)}$ and $b_{(k,n)}$.

$$\left[\sum_{k=1}^{N} a_{(k,n)} e^{-b_{(k,n)}\gamma} \right]^n \tag{14}$$

We reuse Eq. 6 by replacing number of terms N by multinomial terms N_M and get Eq. 15

$$\hat{C} = \left[\frac{C}{B} \right]^n = \log_2^n(1+\gamma) \approxeq \sum_{k=1}^{N_M} a_{(k,n)} e^{-b_{(k,n)}\gamma} \tag{15}$$

The $\kappa - \mu$ distribution is a general fading distribution that can be used to represent the small scale variation of the fading signal in a LOS; it is written in terms of two physical parameters, namely κ and μ with other classical fading included in the $\kappa - \mu$ distribution as particular cases [23].

$$M_{\Omega(\kappa-\mu)}(s) = \left(\frac{\mu(1+\kappa)}{\mu(1+\kappa)+s\Omega} \right)^{\mu} \exp\left(\frac{-\mu\kappa s\Omega}{\mu(1+\kappa)+s\Omega} \right) \tag{16}$$

Shadowing occurs due to obstacles blocking the direct radio path causing slow signal fluctuation. It is introduced in a LOS multipath fading model in two primary ways. The first is based on the assumption that the total power (W) is subjected to random fluctuations, and the second relies on the assumption that only the dominant component is subjected to random fluctuations [24]. Moment generating function of the $\kappa - \mu$ generalized fading distribution is given by Eq. 16 [23] in the closed-form. We consider a receiver equipped with an antenna array with the signals from the L i.n.d diversity paths combined using a coherent MRC rule. The instantaneous SNR at the output of the MRC combiner is simply the sum of the instantaneous received SNRs from all the L i.n.d diversity paths, viz.,

$$\gamma_{MRC} = \sum_{k=1}^{L} \gamma_k$$

Thus the MGF of MRC combiner output SNR is given by Eq. 17 where L is diversity order. The Table I shows the parameters for four branch diversity orders. In special cases, when (κ=7 and μ=1) and (κ=0 and μ=1.5), κ-μ fading distribution becomes Rician and Nakagami-m distribution, respectively.

$$M_{\gamma_{MRC}}(s) = \prod_{k=1}^{L} M_{\gamma_{\kappa-\mu}}(s) \tag{17}$$

The expression for first four moment of channel capacity $\mathbb{E}[C_1]$, $\mathbb{E}[C_2]$, $\mathbb{E}[C_3]$, and $\mathbb{E}[C_4]$ is obtained on substituting n=1,2,3, and 4 respectively in Eq. 22.

TABLE I: Parameters for 4 branch $M_{\gamma_{\kappa-\mu}}(s)$

Kappa κ	= [7;	0;	3;	2];
Mue μ	= [1;	1.50;	2.5;	0.5];
(Diversity Order) wf	= [1;	0.75;	1.5;	1];

C. Calculating channel capacity using Gauss-Hermite quadrature

We present one another method to calculate the HOS of CC using Semi-Infinite Gauss-Hermite Quadrature with 15 terms. By substituting $f_\gamma(\gamma)$ with $f_{\gamma_{(\kappa-\mu)}}(\gamma)$ from Eq. 18 on R.H.S of Eq. 21 we obtain L.H.S of Eq. 21 where, $G(n,b,a)$ is expressed in Eq. 19. Using variable transformation on Eq. 19, let $a\gamma$ = y^2 $\rightarrow d\gamma = \frac{2ydy}{a}$ where, a=$\frac{\mu(1+\kappa)}{\Omega}$ and b=$\frac{\mu-1}{2}$ and using N-point semi-infinite Gauss-Hermite quadrature [25] we obtain Eq. 20.

$$f_{\gamma_{\kappa-\mu}}(\gamma) = \frac{\mu\left(\frac{1+\kappa}{\Omega}\right)^{\frac{\mu+1}{2}} \gamma^{\frac{\mu-1}{2}}}{\kappa^{\frac{\mu-1}{2}} \exp(\mu\kappa)} \exp\left(-\frac{\mu(1+\kappa)\gamma}{\Omega}\right) \times I_{\mu-1}\left(2\mu\sqrt{\frac{\kappa(1+\kappa)\gamma}{\Omega}}\right) \tag{18}$$

$$G\left(n, \frac{\mu-1}{2}, \frac{\mu(1+k)}{\Omega}\right) = \int_0^\infty \log_2^n(1+\gamma)\gamma^{\frac{\mu-1}{2}} \exp\left(-\frac{\mu(1+k)\gamma}{\Omega}\right) \times I_{\mu-1}\left(2\mu\sqrt{\frac{\kappa(1+\kappa)\gamma}{\Omega}}\right) d\gamma \tag{19}$$

where, $G(n,b,a) \approxeq \frac{2}{a^{b+1}} \sum_{k=1}^{15} w_k \log_2^n \left(1+\frac{t_k^2}{a}\right) t_k^{2b+1} \times I_{2b}^{b+1}(2t_k\sqrt{k\mu})$ and values for, w_k and t_k are mentioned in [25]

(20)

$$E\left[\hat{C}_n\right] = \int_0^\infty \log_2^n(1+\gamma) f_\gamma(\gamma) d\gamma = \frac{\mu\left(\frac{1+k}{\Omega}\right)^{\frac{\mu-1}{2}}}{k^{\frac{\mu-1}{2}} \exp^{\mu k}} \int_0^\infty \log_2^n(1+\gamma) f_{\gamma_{\kappa-\mu}}(\gamma) d\gamma = \frac{\mu\left(\frac{1+k}{\Omega}\right)^{\frac{\mu-1}{2}}}{k^{\frac{\mu-1}{2}} \exp^{\mu k}} G\left(n, \frac{\mu-1}{2}, \frac{\mu(1+k)}{\Omega}\right) \tag{21}$$

$$E\left[\hat{C}_n\right] = \int_0^\infty \log_2^n(1+\gamma) f_\gamma(\gamma) d\gamma \approxeq \sum_{k=1}^{N_M} a_{(k,n)} \int_0^\infty e^{-b_{(k,n)}\gamma} f_{\gamma_{\kappa-\mu}} d\gamma \approxeq \sum_{k=1}^{N_M} a_{(k,n)} M_\gamma\left(b_{(k,n)}\right) \tag{22}$$

III. Applications of Exponential - Type Approximations for $\log_2^n(1+\gamma)$ in the Evaluation of HOS

A. On calculating n^{th} moment of Channel Capacity

We observed that if one is only interested in the first and second-order statistics of CC, then the coefficients $a_{(k,n)}$ and $b_{(k,n)}$ obtained for logarithmic function exponential-type approximation only require few terms from N=4 to N=7 and it is quite adequate for getting accurate results using AMSE with no computational complexity. AMSE is much better for our application than RMSE, although RMSE is more preferred over AMSE in general applications. We observe that RMSE emphasizes the smaller SNR region while AMSE shows more accurate fitting in the higher region of SNR. The primary reason we get such trends is that CC is a logarithmic function and logarithmic function is a monotonically increasing function and fitting a sum of exponentials to measured data is generally difficult numerically rather than the ill-conditioned problem of applied data analysis known as the Prony approximation [26]. However, while computing skewness and kurtosis using logarithmic function exponential-type approximation, we faced significant numerical issues in computing the curve-fitting task as it involves solving matrix inversion of the system equations. Also, as the number of terms grows, we find an issue finding the initial seed value and the curve-fitting. One of the ways to address this issue, we find that splitting linear and non-linear terms for n=1 is entirely accurate. With this, we opt to use multinomial expansion to find the coefficients $a_{(k,n)}$ and $b_{(k,n)}$.

To circumvent this challenge, we observed that with n = 1, the curve-fitting is more robust, and we can achieve better approximation using multinomial expansion because it eliminates the ambiguity of curve-fitting the data points. Thus with this, we eliminate the need for coefficients $a_{(k,n)}$ and $b_{(k,n)}$ from the curve-fitting for the logarithmic function exponential-type approximation. The Figure 1 shows how good our approximation is, and Table II shows how many terms are required to perform HOS of CC lucidly.

TABLE II: Summary of minimum terms required to calculate HOS of CC accurately using exponential-type approximation.

Error Metrics ⇓	Number of Terms ⇒	Normalized Ergodic Capacity	Normalized Variance Capacity	Normalized Skewess Capacity	Normalized Kurtosis Capacity
AMSE	N =	4,5,7,10	13,15,17	15,17,19	19
	Mn N =	Not Required	7,8	10,13	15,17
RMSE	N =	4,5,7,10	15,17	17,19	19 not enough
	Mn N =	Not Required	7,8	10,13	15,17

IV. Conclusion

We provide an accurate and efficient technique for computing the desired HOS of CC with different types of diversity combining methods which dramatically improve the accuracy and spectral efficiency of the communication system.

Using this techniques we can evaluate higher-order statistics of error rate and/or channel capacity quite easily which

The performance metrics above-mentioned can efficiently help to select the best channel in heterogeneous wireless networks with spectrum sharing and/or aggregation capabilities. The derived results are novel and also are given in closed form for general fading channels, as opposed to previously published works. The correct expressions for the same are presented in this paper. Thus, the proposed contribution is meaningful, since it addresses to rectify the errors in computing the HOS of the CC over generalized fading channels.

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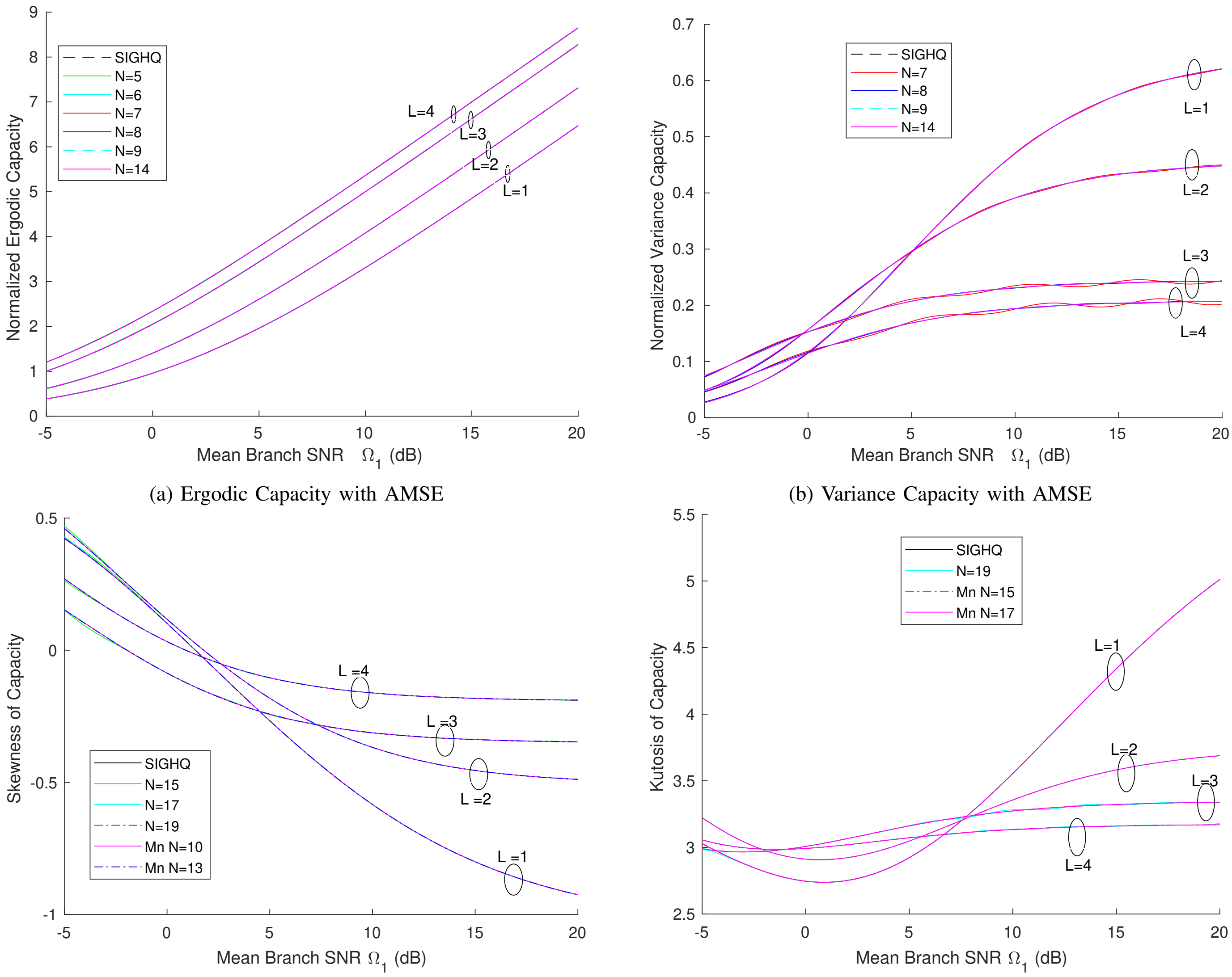

(a) Ergodic Capacity with AMSE

(b) Variance Capacity with AMSE

(c) Skewness Capacity with AMSE

(d) Kurtosis Capacity with AMSE

Fig. 1: HOS of Channel Capacity

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